Neale and the slingshot

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'Slingshot arguments' is a label for a class of arguments which includes Church's argument to the effect that if sentences designate propositions, then there are at most two propositions (1943, pp. 299-300), an argument sketched by Gödel to the effect that all true sentences have the same signification (1983, p. 450), Quine's argument to the effect that quantified modal logic is committed to the view that the modal operators are truth-functional (1966, pp. 160-161), and Davidson's argument to the effect that there is at most one fact (1984, pp. 41-42). These arguments are in a way similar, hence their common label. Of course, their similarity is not due to the fact that they all have the same conclusion. They are similar because they instantiate similar argument-schemata.

In his book *Facing Facts*, Stephen Neale presents two formal proofs which he takes to be abstract versions of Gödel's slingshot and of the slingshots of Church, Quine and Davidson, respectively. From the first he extracts a principle I will call GÖDEL, and from the second a result I will call CHURCH. Both principles state that a one-place sentential connective is truth-functional provided that it satisfies certain substitutivity conditions.

The main philosophical interest of these general principles, as Neale rightly stresses, lies in the fact that they provide one with precise constraints on a wide class of philosophical theories. It is obvious that such principles, if true, impose serious constraints on e.g. theories of modality, theories of belief, and theories of knowledge. For the operators 'it is necessary that', 'it is possible that', 'Sam believes that' and 'Sam knows that' are certainly not truth-functional – at least one should recognise it if one takes them seriously. This is also the case, though not so obviously, for various other theories, like theories of facts, propositions, state of affairs or situations. For instance, a theory of facts which would treat the complex one-place sentential operator 'the fact that 2+2=4 = the fact that …' as truth-functional would arguably be a bad theory. For if that connective is truth-functional, then 'the fact that 2+2=4 = the fact that ϕ' turns out true for absolutely every true sentence ϕ.

On Neale's view, GÖDEL is superior to CHURCH insofar as the conditions which appear in GÖDEL are weaker than those which appear in CHURCH. Neale also takes it that the proof which delivers GÖDEL is superior to the one which delivers CHURCH, insofar as, in order to be correct, the latter, but not the former, has to be supplemented by substantial views about definite descriptions. My main aim in this paper is to reject both of Neale's claims.

1. Inference principles

In this section I introduce five inference principles which will be used later on, following Neale's terminology, when available (see Chapters 7 and 9).[1]

Let © be a one-place sentential connective. We shall say that it obeys the *principle of substitution for material equivalents* - for short, that it is +PSME - iff the following rule of inference is valid:

[1] All references to Neale are to *Facing Facts*.

$$\frac{\begin{array}{c}\phi \equiv \psi \\ ©\Sigma(\phi)\end{array}}{©\Sigma(\psi)}$$

(ϕ and ψ stand for sentences, $\equiv$ for the material equivalence connective, $\Sigma(\phi)$ for an extensional sentence containing ϕ, and $\Sigma(\psi)$ for the result of replacing one or more occurrences of ϕ in $\Sigma(\phi)$ by ψ).

Our connective obeys the *principle of substitution for logical equivalents* - for short, it is +PSLE - iff the following rule of inference is valid:

$$\frac{\begin{array}{c}\phi \Leftrightarrow \psi \\ ©\Sigma(\phi)\end{array}}{©\Sigma(\psi)}$$

($\Leftrightarrow$ is the logical equivalence connective, ϕ, ψ, $\Sigma(\phi)$ and $\Sigma(\psi)$ are as above).

Our connective obeys the *principle of substitution for double negation mates* - for short, it is +DNN - iff the following rules of inference are valid:

$$\frac{©\phi}{©{\sim}{\sim}\phi} \qquad \frac{©{\sim}{\sim}\phi}{©\phi}$$

(ϕ stands for a sentence, and $\sim$ for the negation connective).

If © is +PSME, then it is +PSLE (any logically equivalent sentences are materially equivalent), and if it is +PSLE, then it is +DNN (double negation mates are logically equivalent).

The remaining two inference principles involve definite descriptions. I adopt Neale's convention to the effect that a sentence containing iota-terms is to be understood in such a way that the iota-expressions have minimal scope in that sentence.

Our connective obeys the *principle of iota-substitution* - for short, it is +ι-SUBS - iff the following rules of inference are valid:

$$\frac{\begin{array}{c}\iota x\phi = \iota x\psi \\ ©\Sigma(\iota x\phi)\end{array}}{©\Sigma(\iota x\psi)} \qquad \frac{\begin{array}{c}a = \iota x\phi \\ ©\Sigma(\iota x\phi)\end{array}}{©\Sigma(a)} \qquad \frac{\begin{array}{c}a = \iota x\phi \\ ©\Sigma(a)\end{array}}{©\Sigma(\iota x\phi)}$$

($\iota x\phi$ and $\iota x\psi$ stand for definite descriptions, a for a singular term, $\Sigma(a)$ for an extensional sentence containing some occurrence of a, $\Sigma(\iota x\phi)$ for the result of replacing one or more occurrences of a in $\Sigma(a)$ by $\iota x\phi$, and $\Sigma(\iota x\psi)$ for the result of replacing one or more occurrences of a in $\Sigma(a)$ by $\iota x\psi$).

Finally, our connective obeys the *principle of iota-conversion* - for short, it is +ι-CONV - iff the following rules of inference are valid:

$$\frac{©\mathrm{T}[\Sigma(x/a)]}{©\mathrm{T}[a = \iota x(x{=}a \bullet \Sigma(x))]} \qquad \frac{©\mathrm{T}[a = \iota x(x{=}a \bullet \Sigma(x))]}{©\mathrm{T}[\Sigma(x/a)]}$$

(a stands for a singular term, x for a variable, $\Sigma(x)$ for an extensional formula containing at least one occurrence of x, $\Sigma(x/a)$ for the result of replacing every occurrence of x in $\Sigma(x)$ by a, T[$\Sigma(x/a)$] for an extensional sentence containing $\Sigma(x/a)$, and T[$a = \iota x(x{=}a \bullet \Sigma(x))$] for the result of replacing one or more occurrences of $\Sigma(x/a)$ in T[$\Sigma(x/a)$] by $a = \iota x(x{=}a \bullet \Sigma(x))$).

We shall say that PSME, PSLE, DNN, ι-SUBS or ι-CONV apply to extensional contexts iff the rule applies to the identity truth-functional connective. Note that it is trivially true that PSME, PSLE and DNN apply to extensional contexts, and that under Russell's theory of definite descriptions, so do ι-SUBS and ι-CONV.

2. CHURCH

CHURCH is the following proposition:

(CHURCH) If a one-place connective is both +PSLE and +ι-SUBS, then it is +PSME.

Neale holds that this result can be extracted from the following proof, assuming Russell's theory of definite descriptions (p. 173):

1	[1]	$\phi \equiv \psi$	premise
2	[2]	$©\phi$	premise
2	[3]	$©[a = \iota x((x{=}a \bullet \phi) \vee (x{=}b \bullet {\sim}\phi))]$	2, ©+PSLE
1	[4]	$\iota x((x{=}a \bullet \phi) \vee (x{=}b \bullet {\sim}\phi)) =$ $\iota x((x{=}a \bullet \psi) \vee (x{=}b \bullet {\sim}\psi))$	1, def. of 'ιx'
1,2	[5]	$©[a = \iota x((x{=}a \bullet \psi) \vee (x{=}b \bullet {\sim}\psi))]$	3,4, ©+ι-SUBS
1,2	[6]	$©\psi$	5, ©+PSLE

Unfortunately the proof is faulty under the Russellian treatment of descriptions. Neale claims that by Russell's theory, given any sentence ϕ and any singular terms a and b, ϕ is logically equivalent to $a = \iota x((x{=}a \bullet \phi) \vee (x{=}b \bullet {\sim}\phi))$, but this is not the case. In fact some models with only *one* individual make some sentences false, while given any sentence ϕ and any singular terms a and b, all such models make the Russellian rendering of $a = \iota x((x{=}a \bullet \phi) \vee (x{=}b \bullet {\sim}\phi))$ true.

But a correct proof can be constructed. It is in two parts. The first is given by Neale himself (p. 170), and it establishes that if a one-place connective is both +PSLE and +ι-SUBS, then it permits the substitution *salva veritate* of truths for truths:

1	[1]	ϕ	premise
2	[2]	ψ	premise
3	[3]	$©\phi$	premise
3	[4]	$©(a = \iota x(x{=}a \bullet \phi))$	3, ©+PSLE
1,2	[5]	$\iota x(x{=}a \bullet \phi) = \iota x(x{=}a \bullet \psi)$	1,2, def. of 'ιx'
1,2,3	[6]	$©(a = \iota x(x{=}a \bullet \psi))$	4,5, ©+ι-SUBS
1,2,3	[7]	$©\psi$	6, ©+PSLE

Russell's theory of descriptions justifies the moves to lines [4], [5] and [7]. In order to establish CHURCH, it is then sufficient to establish that if a one-place connective is both +PSLE and +ι-SUBS, then it permits the substitution *salva veritate* of falsehoods for falsehoods. But this is easy given the previous result. For suppose that © is both +PSLE and +ι-SUBS. Then so is the complex connective ©~. Suppose then that ${\sim}\phi$, ${\sim}\psi$. If $©\phi$, then $©{\sim}({\sim}\phi)$ (since © is +PSLE), and so $©{\sim}({\sim}\psi)$ (by the previous result), and so $©\phi$ (since © is +PSLE).

It is clear that less than Russell's theory of descriptions is required for the previous proof to be correct. Say that *definite descriptions are nice* iff the following two conditions are satisfied:

(1) ϕ and $a = \iota x(x{=}a \bullet \phi)$ are logically equivalent;
(2) $\iota x(x{=}a \bullet \phi) = \iota x(x{=}a \bullet \psi)$ is a logical consequence of ϕ, ψ.

Our proof establishes CHURCH on the sole assumption that definite descriptions are nice.

3. GÖDEL

GÖDEL is the following proposition:

(GÖDEL) If a one-place connective is both +ι-CONV and +ι-SUBS, then it is +PSME.

Neale extracts GÖDEL from a long four-part proof (pp. 183-186). The proof actually makes use of an assumption on the connective under consideration which does not appear in GÖDEL, namely the assumption that the connective is +DNN. So at best the proof establishes that:

(GÖDEL) If a one-place connective is +ι-CONV, +ι-SUBS and +DNN, then it is +PSME.

From now on I shall use the label 'GÖDEL' for *this* proposition, and assume that this is the result Neale has in mind.

CHURCH logically follows from the two-part proof given above and the assumption that definite descriptions are nice. Neale's proof for GÖDEL also makes use of an assumption concerning definite descriptions, namely that ι-CONV and ι-SUBS apply to extensional contexts. But GÖDEL does *not* logically follow from Neale's four-part proof and that assumption. What follows is the weaker:

(GÖDEL') If a one-place connective is +ι-CONV, +ι-SUBS and +DNN, then it is +PSME *with respect to subject-predicate sentences*.

Neale is aware of that fact, but he claims, following a suggestion of Gödel's (1983), that a certain assumption yields the more general result – to wit the assumption that "any sentence can be put into subject-predicate form" (see pp. 130 and 186). It is not clear what this means. I suggest, following a hint given by Neale (p. 130), that the principle is that in any context whatsoever, any occurrence of a sentence ϕ can be replaced *salva veritate* by the sentence $|\phi|(a)$ (read, perhaps: 'a is such that ϕ', or: 'ϕ and $a{=}a$') for any (referring) singular term a and *vice versa*. Call this *Gödel's principle*. Using GÖDEL' and Gödel's principle, we can indeed prove GÖDEL. The proof is fairly simple:

1	[1]	$\phi \equiv \psi$	premise
2	[2]	$©\phi$	premise
1	[3]	$\lvert\phi\rvert(a) \equiv \lvert\psi\rvert(b)$	1, Gödel's principle
2	[4]	$©\lvert\phi\rvert(a)$	2, Gödel's principle
1,2	[5]	$©\lvert\psi\rvert(b)$	3,4, GÖDEL'
1,2	[6]	$©\psi$	5, Gödel's principle

Note that Gödel's principle, as stated, is doubtful. Think about the connectives 'the proposition that ... is about Socrates' and 'the proposition that ... is about nothing'. Some restriction on sentential contexts is needed, but here I shall ignore the issue.

Thus Neale's abstract version of Gödel's slingshot establishes that GÖDEL' is true if ι-CONV and ι-SUBS apply to extensional contexts, and from this result and Gödel's principle we get the conclusion that GÖDEL is true if ι-CONV and ι-SUBS apply to such contexts. Now I wish to show that if Gödel's principle is accepted, then the very same conclusion can be drawn directly from a fairly simple formal proof, a proof which is actually simpler than Neale's. The proof is in two parts. The first is the following:

1	[1]	ϕ	premise
2	[2]	ψ	premise
3	[3]	$©\phi$	premise
1	[4]	$\vert\phi\vert(a)$	1, Gödel's principle
2	[5]	$\vert\psi\vert(a)$	2, Gödel's principle
3	[6]	$©\vert\phi\vert(a)$	3, Gödel's principle
1	[7]	$a = \iota x(x=a \bullet \vert\phi\vert(x))$	4, ι-CONV
2	[8]	$a = \iota x(x=a \bullet \vert\psi\vert(x))$	5, ι-CONV
3	[9]	$©(a = \iota x(x=a \bullet \vert\phi\vert(x)))$	6, ©+ι-CONV
1,3	[10]	$©(a = a)$	7,9, ©+ι-SUBS
1,2,3	[11]	$©(a = \iota x(x=a \bullet \vert\psi\vert(x)))$	8,10, ©+ι-SUBS
1,2,3	[12]	$©\vert\psi\vert(a)$	11, ©+ι-CONV
1,2,3	[13]	$©\psi$	12, Gödel's principle

That proof establishes that if a one-place connective is +ι-CONV and +ι-SUBS, then it permits the substitution *salva veritate* of truths for truths. In order to establish GÖDEL', it is then sufficient to establish that if a one-place connective is both is +ι-CONV, +ι-SUBS and +DNN, then it permits the substitution *salva veritate* of falsehoods for falsehoods. But this is easy given the previous result. For suppose that © is +ι-CONV, +ι-SUBS and +DNN. Then in particular the complex connective ©~ is +ι-CONV and +ι-SUBS. Suppose now that $\sim\phi$, $\sim\psi$. If $©\phi$, then $©\sim(\sim\phi)$ (since © is +DNN), and so $©\sim(\sim\psi)$ (by the previous result), and so $©\phi$ (since © is +DNN).

Note that this proof establishes that GÖDEL is true if ι-CONV *alone* applies to extensional contexts: the assumption that ι-SUBS applies to such contexts is not used. (It is also possible to modify Neale's proof so as to establish that GÖDEL' is true if ι-CONV alone applies to extensional contexts.)

4. CHURCH VS. GÖDEL

So our formal proof for CHURCH establishes that

(CHURCH) If a one-place connective is both +PSLE and +ι-SUBS, then it is +PSME.

on the sole assumption that definite descriptions are nice, and our formal proof for GÖDEL establishes that

(GÖDEL) If a one-place connective is +ι-CONV, +ι-SUBS and +DNN, then it is +PSME.

on the sole assumption that ι-CONV applies to extensional contexts (and that Gödel's principle is correct). Given that on a Russellian account of definite descriptions, definite descriptions are nice and ι-CONV applies to extensional contexts, on that account both CHURCH and (taking Gödel's principle for granted) GÖDEL hold.

Neale holds that his proof for GÖDEL is superior to his proof for CHURCH because the latter, but not the former, involves some moves whose justification requires some substantial, semantic assumptions about definite descriptions (see pp. 171 and 184-185). And he also claims that on a Russellian treatment of descriptions, GÖDEL is in a certain sense more worrying than CHURCH (see p. 201).

I think both claims are false.

(1) In the proof for CHURCH (forget Neale's faulty proof), we have made use of the sole extra assumption that definite descriptions are nice. On the other hand, Neale makes certain assumptions about definite descriptions in his proof for GÖDEL, namely that both ι-CONV and ι-SUBS apply to extensional contexts. Now is the view that definite descriptions are nice more substantial than the view that ι-CONV and ι-SUBS apply to extensional contexts? I don't think so.

For the view that definite descriptions are nice follows from (i) the view that both ι-CONV and ι-SUBS apply to extensional contexts and (ii) extremely weak, non-substantial principles. These principles are *clone-elimination* and *the-a*. Clone-elimination says that $a = \iota x(x=a \bullet \phi \bullet x=a)$ entails $a = \iota x(x=a \bullet \phi)$. The-a says that "the *F* is an *F*", more precisely, that $a = \iota x\Sigma(x)$ entails $\Sigma(a)$. The-a ensures that ϕ is a logical consequence of $a = \iota x(x=a \bullet \phi)$. In fact:

1	[1]	$a = \iota x(x=a \bullet \phi)$	premise
1	[2]	$a=a \bullet \phi$	1, the-a
1	[3]	ϕ	2, conjunction-elimination

Clone-elimination and the view that ι-CONV applies to extensional contexts guarantee that $a = \iota x(x=a \bullet \phi)$ is a logical consequence of ϕ:

1	[1]	ϕ	premise
	[2]	$a=a$	classical logic
1	[3]	$\phi \bullet a=a$	1,2, conjunction-introduction
1	[4]	$a = \iota x(x=a \bullet \phi \bullet x=a)$	3, ι-CONV
1	[5]	$a = \iota x(x=a \bullet \phi)$	4, clone-elimination

The previous result plus the view that ι-SUBS applies to extensional contexts ensure that $\iota x(x=a \bullet \phi) = \iota x(x=a \bullet \psi)$ is a logical consequence of ϕ, ψ:

1	[1]	ϕ	premise
2	[2]	ψ	premise
1	[3]	$a = \iota x(x=a \bullet \phi)$	1, previous result
2	[4]	$a = \iota x(x=a \bullet \psi)$	2, previous result
1,2	[5]	$\iota x(x=a \bullet \phi) = \iota x(x=a \bullet \psi)$	3,4, ι-SUBS

My proof for GÖDEL does not make use of the assumption that ι-SUBS applies to extensional contexts, it makes use only of the assumption that ι-CONV applies to such contexts, and as I previously emphasized, Neale's proof can easily be modified so that also only this assumption is used. Is the view that definite descriptions are nice more substantial than the view that ι-CONV applies to extensional contexts? Again, I do not think so. For the view that definite descriptions are nice follows from the view that ι-CONV applies to extensional contexts, plus clone-elimination, plus the-a, plus another principle I shall call *fusion*, which says that $a = \iota x(x{=}a \bullet \phi)$ and $a = \iota x(x{=}a \bullet \psi)$ together entail $\iota x(x{=}a \bullet \phi) = \iota x(x{=}a \bullet \psi)$. And in my opinion, fusion is like clone-elimination and the-a, to wit far from being substantial.

(2) Let me now turn to Neale's second claim. The claim is that on a Russellian treatment of descriptions, GÖDEL is more worrying than CHURCH "on the obvious assumption that every "Gödelian equivalence", as given by ι-CONV, is also a logical equivalence, but not *vice versa*" (p. 201). The idea seems to be the following. Let **C** be the class of +PSLE, +ι-SUBS connectives, and let **G** be the class of +ι-CONV, +ι-SUBS, +DNN connectives. CHURCH states that every connective in **C** is truth-functional, and GÖDEL that every connective in **G** is truth-functional. Neale seems to think that assuming Russell's theory of descriptions, **C** is a proper part of **G**, and that it is for that reason that GÖDEL is more worrying than CHURCH on Russell's account. We may agree that *if* **C** is a proper part of **G**, then GÖDEL is indeed more worrying than CHURCH: both principles say that a connective is truth-functional if it satisfies certain conditions, and that **C** is a proper part of **G** means that the GÖDEL-conditions are strictly weaker than the CHURCH-conditions.

But under Russell's theory, **C** is *not* a proper part of **G**. For let us assume that theory (and Gödel's Principle, which is not discussed here). Then the following three conditions are equivalent:

(1) +ι-SUBS +ι-CONV +DNN
(2) +ι-SUBS +PSLE
(3) +PSME

In fact, under Russell's theory, both CHURCH and GÖDEL are true, i.e. (2) entails (3) and (1) entails (3). On the other hand, if a connective is +PSME, it is +PSLE (two logically equivalent sentences are materially equivalent), and under Russell's theory, if a connective is +PSME, it is +ι-SUBS (if $\iota x\phi = \iota x\psi$, then $\Sigma(\iota x\phi)$ and $\Sigma(\iota x\psi)$ are materially equivalent, etc.). Thus (3) entails (2). Finally, double-negation mates are logically equivalent, and as Neale says, on a Russellian account of definite descriptions, every "Gödelian equivalence", as given by ι-CONV, is also a logical equivalence. So (2) entails (1). Now given that (1) and (2) entail each other, **C** = **G**, and GÖDEL and CHURCH have the same "worrying character". They are indeed the same result: their consequents are identical, and their antecedent equivalent.

Let me here in passing answer a question Neale raises at the end of his Chapter on GÖDEL (p. 187). The question, it seems to me, is whether, under Russell's theory of descriptions, there is a condition F on one-place connectives such that a connective's being +F and +ι-SUBS entails its being +PSME, which is such that the conjunctive condition +F +ι-SUBS is not stronger (e.g. strictly weaker) than the condition in GÖDEL. The answer is 'no', for as we just saw, (3) entails (1).

So under Russell's theory of definite descriptions, both CHURCH and GÖDEL (assuming Gödel's principle) are true. What if another view about descriptions is countenanced? I take it that any sensible theory of descriptions should make the principles clone-elimination, the-a and fusion correct. I also take it that any sensible theory of descriptions should validate a further principle, *mini-ι-CONV*, which may be formulated as follows:

$$\frac{\Sigma(a)}{a = \iota x(x{=}a \bullet \Sigma(x))}$$

(a stands for a singular term, x for a variable, $\Sigma(x)$ for an extensional formula containing at least one occurrence of x, and $\Sigma(x/a)$ for the result of replacing every occurrence of x in $\Sigma(x)$ by a). Now from these four principles, it follows that definite descriptions are nice, and so that CHURCH is true (see the last three proofs above). Reflecting on Neale's proof for GÖDEL', one may realize that in fact we need less than the assumption that ι-CONV and ι-SUBS apply to extensional contexts: it is enough to assume that mini-ι-CONV and fusion hold (this is obvious from the text). So by these two rules, GÖDEL' holds, and so by Gödel's principle, GÖDEL too. Reflecting on my own proof for GÖDEL, one will immediately realize that we need less than the assumption that ι-CONV applies to extensional contexts: it is enough to assume that mini-ι-CONV is valid. Thus, calling *minimality* the conjunction of fusion, clone-elimination and the-a, the following two principles hold unconditionally:

(full-CHURCH) If mini-ι-CONV and minimality are correct, then CHURCH.

(full-GÖDEL) If mini-ι-CONV and Gödel's principle are correct, then GÖDEL.

And so my view is that any sensible theory of descriptions should ensure the truth of CHURCH, and the truth of GÖDEL conditional upon Gödel's principle. If one thinks that CHURCH and GÖDEL are true and also that any acceptable theory of descriptions should have it that ι-CONV and +ι-SUBS apply to extensional contexts - which is Neale's case - then one will have to accept that the three sets of conditions mentioned above:

(1) +ι-SUBS +ι-CONV +DNN
(2) +ι-SUBS +PSLE
(3) +PSME

are equivalent, and in particular that CHURCH and GÖDEL are essentially the same result.

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